Existence and Uniqueness of Load-Flow Solutions in Three-Phase Distribution Networks

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***Abstract*—We present sufficient conditions for the existence and uniqueness of load-flow solutions in three-phase distribution networks. The conditions can be efficiently verified for real distribution systems.**

***Index Terms*—load flow solution, fixed point method, existence and uniqueness, distribution networks.**

I. Introduction

In distribution networks, many control procedures rely on the existence and uniqueness of load-flow solutions. However, due to the non-linearity of the load-flow equations, these properties are difficult to study. In this letter, we give efficiently verifiable conditions that guarantee the existence and uniqueness of load-flow solutions in three-phase distribution networks.

II. Problem Formulation

We consider a three-phase network that has one slack bus indexed by 0, N PQ buses indexed by $1,...,N$, and a generic topology (i.e., radial or meshed). Let $\mathbf{v}_j \triangleq (v_j^a, v_j^b, v_j^c)^T$, $\mathbf{s}_j \triangleq (s_j^a, s_j^b, s_j^c)^T$ be the complex vectors representing three-phase nodal voltage and power injection at bus $j \in \{0,...,N\}$, and define $\mathbf{v} \triangleq (\mathbf{v}_1^T,...,\mathbf{v}_N^T)^T$, $\mathbf{s} \triangleq (\mathbf{s}_1^T,...,\mathbf{s}_N^T)^T$. Complex conjugates are denoted by adding an overline $^-$. Then, the load-flow problem consists in solving for $\mathbf{v}$ and $\mathbf{s}_0$ in the following equations, where $\mathbf{s}$ and $\mathbf{v}_0$ are given and $\mathbf{Y}$ is the three-phase compound admittance matrix [1]:

$$\begin{bmatrix} \mathbf{s}_0 \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathrm{diag}(\mathbf{v}_0) & \\ & \mathrm{diag}(\mathbf{v}) \end{bmatrix} \overline{\mathbf{Y}} \begin{bmatrix} \overline{\mathbf{v}}_0 \\ \overline{\mathbf{v}} \end{bmatrix}$$

Notice that matrix $\mathbf{Y}$ can be partitioned as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{00} & \mathbf{Y}_{0L} \\ \mathbf{Y}_{L0} & \mathbf{Y}_{LL} \end{bmatrix}$$

with 3×3 matrix $\mathbf{Y}_{00}$, $3 \times 3N$ matrix $\mathbf{Y}_{0L}$, $3N \times 3$ matrix $\mathbf{Y}_{L0}$, and $3N \times 3N$ matrix $\mathbf{Y}_{LL}$ (see Section IV-C for its invertibility). Then by defining the zero-load voltage $\mathbf{w} \triangleq -\mathbf{Y}_{LL}^{-1}\mathbf{Y}_{L0}\mathbf{v}_0 = (w_1^a, w_1^b, w_1^c, ..., w_N^a, w_N^b, w_N^c)^T$, the load-flow problem can be reduced to Eq.(1), where $\mathbf{v}$ is unknown and $\mathbf{s}$ is given :

$$\mathbf{v} = \mathbf{w} + \mathbf{Y}_{LL}^{-1}\mathrm{diag}(\overline{\mathbf{v}})^{-1}\overline{\mathbf{s}} \tag{1}$$

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Eq.(1) is called the implicit Z_{bus} formulation [2]; it is a fixed point equation in $\mathbf{v}$ and can be solved by the iterative scheme in Eq.(2) :

$$\mathbf{v}^{(k+1)} = \mathbf{w} + \mathbf{Y}_{LL}^{-1}\mathrm{diag}(\overline{\mathbf{v}}^{(k)})^{-1}\overline{\mathbf{s}} \tag{2}$$

In the rest of this letter, we give sufficient conditions under which there exists a unique load-flow solution that can be found by the iteration in Eq.(2).

III. Results

In Theorem 1 we give a result assuming that we have prior knowledge of one pair $(\hat{\mathbf{v}}, \hat{\mathbf{s}})$ that satisfies Eq.(1). This is true in the cases where the load-flow problem is repeatedly solved for varying operational conditions (namely, $\hat{\mathbf{v}}$ is a solution computed in a previous instance of the problem when the power injection was $\hat{\mathbf{s}}$), or where a pair $(\hat{\mathbf{v}}, \hat{\mathbf{s}})$ is obtained by other methods. In Corollary 1, we give a result that does not depend on such prior knowledge.

The result makes use of the following notation :

$$\xi(\mathbf{s}) \triangleq \|\mathbf{W}^{-1}\mathbf{Y}_{LL}^{-1}\overline{\mathbf{W}}^{-1}\mathrm{diag}(\overline{\mathbf{s}})\|_\infty$$

$$u_{\min}(\mathbf{v}) \triangleq \min_{j\in\{1,...,N\},\gamma\in\{a,b,c\}} |v_j^\gamma / w_j^\gamma|$$

$$\mathcal{D}(\rho, \hat{\mathbf{v}}) \triangleq \{\mathbf{v} : |v_j^\gamma - \hat{v}_j^\gamma| \le \rho |w_j^\gamma|, j \in \{1,...,N\}, \gamma \in \{a,b,c\}\}.$$

In the above, $\mathbf{W} \triangleq \mathrm{diag}(\mathbf{w})$, and $\|\bullet\|_\infty$ is the matrix norm induced by the ℓ^∞ norm (i.e., $\|A\|_\infty = \max_{1\le j\le 3N}\sum_{k=1}^{3N}|A_{j,k}|$).

Theorem 1. *Let $\hat{\mathbf{v}}$ be a solution to Eq.(1) with power injection $\hat{\mathbf{s}}$ and assume that $(u_{\min}(\hat{\mathbf{v}}))^2 > \xi(\hat{\mathbf{s}})$.*

For any other power injection vector $\mathbf{s}$ that satisfies

$$\xi(\mathbf{s} - \hat{\mathbf{s}}) < \tfrac{1}{4}\left(u_{\min}(\hat{\mathbf{v}}) - \xi(\hat{\mathbf{s}})/u_{\min}(\hat{\mathbf{v}})\right)^2$$

there is a unique solution $\mathbf{v}$ in $\mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$ to Eq.(1) with power injection $\mathbf{s}$, where $\rho^\ddagger \triangleq \frac{1}{2}\left(u_{\min}(\hat{\mathbf{v}}) - \xi(\hat{\mathbf{s}})/u_{\min}(\hat{\mathbf{v}})\right)$.

This unique solution can be reached by applying the iteration in Eq.(2) initialized with any $\mathbf{v}^{(0)} \in \mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$.

Moreover, this unique solution is located in the smaller domain $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$ with

$$\rho^\dagger \triangleq \rho^\ddagger - \tfrac{1}{2}\sqrt{\left(u_{\min}(\hat{\mathbf{v}}) - \xi(\hat{\mathbf{s}})/u_{\min}(\hat{\mathbf{v}})\right)^2 - 4\xi(\mathbf{s} - \hat{\mathbf{s}})}.$$

Observe that the localization in the smaller domain $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$ is more accurate but depends (via $\rho^\dagger$) on the specific $\mathbf{s}$, unlike the larger domain $\mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$. Also note that the theorem implies that there is no solution $\mathbf{v}$ in $\mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}}) \setminus \mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$.

If no $(\hat{\mathbf{v}}, \hat{\mathbf{s}})$ is obtained, we can use the following corollary.

Corollary 1. *Suppose that the power injection* $\mathbf{s}$ *satisfies* $\xi(\mathbf{s}) < 0.25$. *There exists a unique solution* $\mathbf{v}$ *in* $\mathcal{D}(0.5, \mathbf{w})$ *to Eq.(1) with power injection* $\mathbf{s}$. *This solution can be reached by applying the iteration in Eq.(2) initialized with any* $\mathbf{v}^{(0)} \in \mathcal{D}(0.5, \mathbf{w})$. *Moreover, it is located in the smaller domain* $\mathcal{D}(\rho, \mathbf{w})$ *with* $\rho = (1 - \sqrt{1 - 4\xi(\mathbf{s})})/2$.

Note that all the proposed conditions on $\hat{\mathbf{v}}, \hat{\mathbf{s}}, \mathbf{s}$ can be verified at low computational complexity prior to solving a load-flow problem. Thus, they can be used for network control applications like Distribution Management Systems that need to solve multiple instances of load-flow problems in real time.

Remark 1. Theorem 1 and Corollary 1 are essentially extension to general three-phase distribution networks of the main results for single-phase networks in [3]. The key steps of this extension are (i) formulating the three-phase load-flow problem in the same algebraic form as its single-phase counterpart, and (ii) proving the invertibility of matrix $\mathbf{Y}_{LL}$ in three-phase networks.

IV. Proofs

A. Proof of Theorem 1

Proof. The three-phase implicit Z_{bus} formulation in Eq.(1) has exactly the same algebraic form as its single-phase counterpart in [3], thus we can directly follow and apply the proof of Lemma 1 and Lemma 2 in [3]. Let $\mathbf{G}(\mathbf{v})$ express the right-handside of Eq.(1). It follows that, whenever the conditions in Theorem 1 are true, $\mathbf{G}()$ is a self-mapping and contraction mapping on $\mathcal{D}(\rho, \hat{\mathbf{v}})$ for any $\rho \in [\rho^\dagger, \rho^\ddagger]$. Therefore, according to Banach's fixed point theorem [4], there is a unique solution in $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$ to Eq.(1), and it can be reached by the iteration in Eq.(2) for any $\mathbf{v}^{(0)} \in \mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$; the same also holds if we replace $\rho^\dagger$ by $\rho^\ddagger$. Since $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}}) \subseteq \mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$, we conclude that (i) there is a unique solution in $\mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$; (ii) the solution is located in $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$; (iii) it can be reached by iteration in Eq.(2) with any $\mathbf{v}^{(0)} \in \mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$. □

B. Proof of Corollary 1

Proof. Apply Theorem 1 with $\hat{\mathbf{v}} = \mathbf{w}$ and $\hat{\mathbf{s}} = \mathbf{0}$. □

C. Invertibility of $\mathbf{Y}_{LL}$

The admittance matrix depends on specific device modeling. Here, we show that $\mathbf{Y}_{LL}$ is invertible when the following system assumptions hold, which covers most practical cases.

- The longitudinal component and shunt elements of transmission line between any pair of buses are described by circulant matrices;
- Transformer between any pair of buses is depicted by models in [5], but equipped with complex ratio;
- The connection from the slack bus to a PQ bus can be realized via (i) a transmission line, or (ii) a transformer of either $\mathrm{Delta} - \mathrm{Wye_G}$ or $\mathrm{Wye_G} - \mathrm{Wye_G}$ configuration;
- The connection between two PQ buses can be established through either a transmission line or a transformer of $\mathrm{Wye_G} - \mathrm{Wye_G}$ configuration;
- Transformers may have additional core losses in the form of transverse components attached to related buses;
- Transmission lines and transformers do not generate active power, and their longitudinal components have positive resistance in zero-, positive-, negative sequences.

Proof. To prove that $\mathbf{Y}_{LL}$ is invertible we show that if $\mathbf{x} \in \mathbb{C}^{3N}$ is a vector such that $\mathbf{Y}_{LL}\mathbf{x} = 0$ then we must have $\mathbf{x} = \mathbf{0}$. We can view $\mathbf{Y}_{LL}$ as the admittance matrix of a fictitious N-bus network (i.e., the original network with the slack bus grounded) and $\mathbf{x}$ as its nodal voltage vector; then the sum of all nodal power injections of this N-bus network is $s^{\mathrm{total}} = \mathbf{x}^T \overline{\mathbf{Y}}_{LL} \overline{\mathbf{x}} = 0$.

Let $\mathcal{V}^{\mathrm{slack}}$ be the set of buses that are connected with the slack bus in the original $N+1$-bus network, and write $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_1^T, ..., \mathbf{x}_N^T)^T$ with $\mathbf{x}_j, j \in \{1, ..., N\}$ interpreted as the three-phase nodal voltage at bus j. Note that s^{total} equals the system power loss and can be further decomposed as

$$s^{\mathrm{total}} = s^{\mathrm{slack}} + s^{\mathrm{line}} + s^{\mathrm{shunt}} + s^{\mathrm{leakage}} + s^{\mathrm{core-loss}}$$

where

- s^{slack} is the power consumption in the transverse components that result from connections to the slack bus in the original $N+1$-bus network of all buses $j \in \mathcal{V}^{\mathrm{slack}}$;
- s^{line} is the power consumption in the longitudinal components of all transmission lines;
- s^{shunt} is the power consumption in all shunt elements;
- s^{leakage} is the power consumption that results from all transformer leakage impedances;
- $s^{\mathrm{core-loss}}$ is the power consumption caused by core losses in all transformers.

By the last item of assumption, $s^{\mathrm{total}} = 0$ implies that all the five terms contain zero real parts. Therefore, according to the system assumptions, we have

1) $\Re(s^{\mathrm{slack}}) = 0$ implies that $\mathbf{x}_j = \mathbf{0}$ for all $j \in \mathcal{V}^{\mathrm{slack}}$;
2) From $\Re(s^{\mathrm{line}}) = 0$, it can be obtained that $\mathbf{x}_j = \mathbf{x}_l$ for transmission line between any buses $j, l \in \{1, ..., N\}$;
3) By $\Re(s^{\mathrm{leakage}}) = 0$ and the $\mathrm{Wye_G} - \mathrm{Wye_G}$ configuration, we have $\mathbf{x}_j = K_{jl}\mathbf{x}_l$ for transformer with ratio K_{jl} between any buses $j, l \in \{1, ..., N\}$.

From the first item, there is at least one bus that has zero voltage. From the second and the third items, the zero voltage propagates throughout the N-bus network. Thus, we have $\mathbf{x} = \mathbf{0}$. □

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